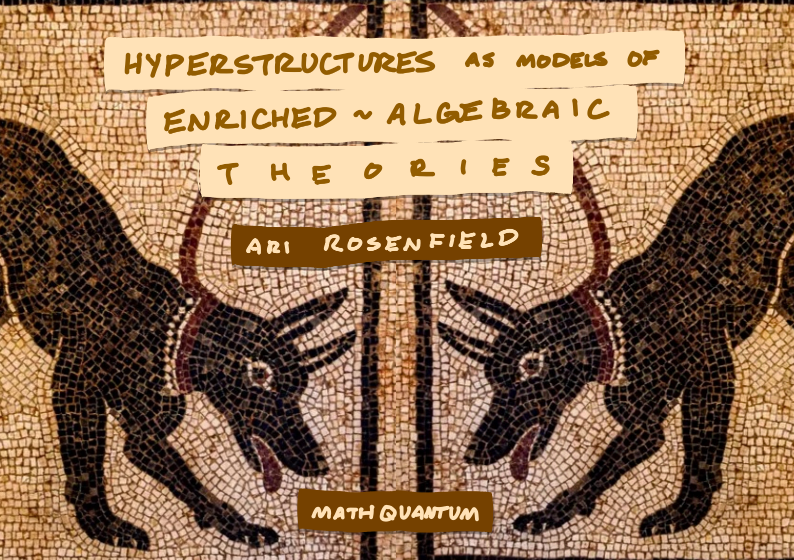


HYPERSTRUCTURES AS MODELS OF ENRICHED ~ ALGEBRAIC THEORIES

ARI ROSENFIELD

MATH QUANTUM

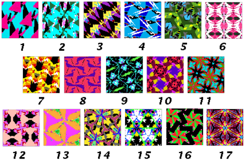
WHAT ARE HYPERSTRUCTURES?

WE KNOW AND LOVE ALGEBRAIC STRUCTURES.

e.g.,

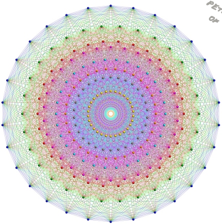
GROUPS

WALLPAPER GRPS.



RINGS

PETRIE PROJECTION OF A LIE ALGEBRA



THEY'RE DEFINED IN TERMS OF BINARY OPERATIONS,

FUNCTIONS $G \times G \longrightarrow G$.

$(a, b) \longmapsto a + b$ 2 INPUTS, 1 OUTPUT.

WHAT ARE HYPERSTRUCTURES?

ONE COULD ASK: "WHAT KIND OF MATHEMATICAL THING DO I GET IF I ALLOW THE ADDITION ON MY MONOID/GROUP/RING/FIELD TO BE MULTI-VALUED?"

IN OTHER WORDS: "WHAT IF MY 'ADDITION LAW' HAS THE FORM

$$G \times G \longrightarrow \mathcal{P}(G)$$

INSTEAD OF $G \times G \longrightarrow G$?"

WHAT ARE HYPERSTRUCTURES?

OKAY, ONE COULD ASK THAT, BUT WHY WOULD THEY? FIRST, TERMINOLOGY:

A SET X EQUIPPED WITH A FUNCTION $X \times X \xrightarrow{*} \mathcal{P}(X)$ IS CALLED A HYPERMAGMA.

WHAT ARE HYPERSTRUCTURES?

e.g., A HYPERMONOID IS AN ASSOCIATIVE, UNITAL HYPERMAGMA.

ANY MONOID $(X, +, e)$ IS A HYPERMONOID VIA $x \star y := \{x+y\}$, $\mathbb{1} := \{e\}$.

WHAT ARE HYPERSTRUCTURES?

e.g., A MOSAIC IS A NON-ASSOCIATIVE, UNITAL REVERSIBLE HYPERMAGMA.

"CORRECT" NOTION OF INVERSES

IF G IS A GROUP AND $H \leq G$, THE DOUBLE COSETS HgH ARE A MOSAIC WITH

$$HgH \star HhH := \{HjH : j \in HgHhH\}.$$

$$\mathbb{1} := He_G H, \quad (HgH)^{-1} := Hg^{-1}H.$$

WHAT ARE HYPERSTRUCTURES?

GENERIC "STRUCTURED" HYPERMAGMA: WILL SAY HYPERSTRUCTURE.

TURNS OUT THAT THESE APPEAR NATURALLY WHEN STUDYING NUMBER THEORY, REPRESENTATION THEORY, QUANTUM ERROR CORRECTION:

HYPERSTRUCTURES IN THE WILD

Journal of Number Theory
Volume 131, Issue 2, February 2011, Pages 159-194

The hyperring of adèle classes

Alain Connes [a b c], Caterina Consani [d]

IDEA: DO ALGEBRAIC GEOMETRY OVER "$\mathbb{F}_1$" TOWARD A PROOF OF THE RIEMANN HYPOTHESIS — THE KRASNER HYPERFIELD

$\mathbb{K} = \{0,1\}$ $\longleftrightarrow$ $0+0=\{0\}$, $1+0=0+1=\{1\}$, $1+1=\{0,1\}$

ENDS UP BEING A GOOD CANDIDATE FOR "$\mathbb{F}_1$" IN THAT $\mathrm{Hom}_{HRing}(X, \mathbb{K}) \cong \mathrm{Spec}\, X$.

HYPERSTRUCTURES IN THE WILD

IDEA: DO REPRESENTATION THEORY OF COMPACT

LEONID VAINERMAN

Hypergroup structures associated with Gel'fand pairs of compact quantum groups

Astérisque, tome 232 (1995), p. 231-242

QUANTUM GROUPS — ANY GEL'FAND PAIR

$$(G, H), \quad H \leq G$$

GIVES RISE TO A COMMUTATIVE HYPERGROUP WHOSE UNDERLYING SET IS THE DOUBLE COSETS OF H IN G.

HYPERSTRUCTURES IN THE WILD

IDEA: CHARACTERIZE CSS CODE STATES AS A CLASS OF MATROIDS

Local equivalence, surface-code states, and matroids

Pradeep Sarvepalli* and Robert Raussendorf

Show more

Phys. Rev. A **82**, 022304 – **Published 5 August, 2010**

ELSEVIER

Journal of Algebra
Volume 676, 15 August 2025, Pages 408-474

Research Paper

Categories of hypermagmas, hypergroups, and related hyperstructures ☆

So Nakamura, Manuel L. Reyes

FACT: ANY MATROID CAN BE REALIZED CANONICALLY AS A MOSAIC.

THE POINT

HYPERSTRUCTURES ARE WORTH STUDYING IF YOU CARE ABOUT QUANTUM ALGEBRA AND/OR QUANTUM ERROR CORRECTION.

THE PROBLEM

ME, A CATEGORY THEORIST

ALL OF MATHEMATICS

GIVEN A CAT. $\mathcal{H}$ OF HYPERSTRUCTURES, WHAT CONSTRUCTIONS CAN BE DONE IN $\mathcal{H}$?

IS IT BICOMPLETE? MONOIDAL? SYMMETRIC? CLOSED? ABELIAN? ...

THE PROBLEM

ELSEVIER

Journal of Algebra
Volume 676, 15 August 2025, Pages 408-474

Research Paper

Categories of hypermagmas, hypergroups, and related hyperstructures ☆

So Nakamura, Manuel L. Reyes

HMag, HSemi, HMon, Msc are all bicomplete and symmetric monoidal closed.
HGrp is not!

LABORIOUS, CONSTRUCTIVE PROOFS.

IN A GOOD WAY! ♡

THE PROBLEM

FOR $\mathbf{Grp}$, $\mathbf{Ab}$, $R\mathbf{Mod}$, $\mathbf{LieAlg}_K$, ETC., BICOMPLETENESS, MONOIDAL CLOSED STRUCTURE, SYMMETRY FOLLOW FROM THESE BEING ALGEBRAIC CATEGORIES.

WHAT'S THAT?

UNIVERSAL ALGEBRA

AN ALGEBRAIC THEORY IS A CATEGORY WHOSE OBJECTS ALL HAVE THE FORM

CATEGORICAL PRODUCT $\quad \bullet \times \bullet \times \dots \times \bullet =: \bullet^n$

FOR SOME DISTINGUISHED OBJECT $\bullet$.

MORPHISMS $\bullet^n \longrightarrow \bullet$ ARE THEN n-ARY OPERATIONS DEFINED ON $\bullet$.

"BLUEPRINT FOR A GIVEN TYPE OF ALGEBRAIC STRUCTURE."

UNIVERSAL ALGEBRA

A MODEL OF AN ALGEBRAIC THEORY $\mathcal{T}$ IS A FUNCTOR $\mathcal{T} \longrightarrow \mathrm{Set}$ WHICH PRESERVES FINITE PRODUCTS.

" TAKE YOUR BLUEPRINT AND BUILD IT."

AN ALGEBRAIC CATEGORY IS ONE WHICH IS EQUIVALENT TO THE CATEGORY OF MODELS FOR SOME ALGEBRAIC THEORY.

... IN PARTICULAR, $\mathrm{Mod}\,\mathcal{T} \subseteq [\mathcal{T}, \mathrm{Set}]$.

UNIVERSAL ALGEBRA

e.g., (THEORY OF ABELIAN GROUPS)

$\mathcal{T}_{Ab}$ HAS $\mathbb{N}$ AS OBJECTS AND

$$\mathcal{T}_{Ab}(n,k) = Mat_{k\times n}(\mathbb{Z}).$$

ANY $G \in Ab$ DEFINES A FUNCTOR

$$n \longmapsto G \times \dots \times G \quad (n \text{ TIMES})$$

$$A \in Mat_{k\times n} \longmapsto \left(\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \longmapsto A \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right).$$

(PRODUCT PRESERVING: $n \times k = n + k$.)

UNIVERSAL ALGEBRA

IN THE OPPOSITE DIRECTION, GIVEN $F : \mathcal{T}_{Ab} \to \mathbf{Set}$, SET $G := F(1)$.

DENOTE $m := \begin{bmatrix} 1 & 1 \end{bmatrix} : 2 \to 1$

$\sigma := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : 2 \to 2$, $\rho := \begin{bmatrix} 1 \\ 0 \end{bmatrix} : 1 \to 2$.

THE AXIOMS OF AN ABELIAN GROUP FOLLOW FROM FUNCTORIALITY —

$(F(AB) = F(A)F(B))$

UNIVERSAL ALGEBRA

FOR EXAMPLE (RIGHT UNITALITY)

$$m\rho = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix},$$

SO $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} g \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} g \\ 0 \end{bmatrix} = g + 0$

$$= g = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} g \end{bmatrix}$$

AND (ABELIAN-NESS)

$$m\sigma = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} = m,$$

SO $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix} = h + g = g + h = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}.$

UNIVERSAL ALGEBRA

FOR $\mathcal{T}$ A THEORY, $\mathrm{Mod}\,\mathcal{T}$:

- ADMITS A SYMMETRIC MONOIDAL CLOSED STRUCTURE VIA DAY CONVOLUTION
- IS COMPLETE AND COCOMPLETE.

THIS WHOLE SETUP GENERALIZES TO CATEGORIES & FUNCTORS ENRICHED OVER $\mathcal{V} \in \mathrm{SymMonCl}$.

THE PROBLEM

HYPERSTRUCTURES ARE ALGEBRAIC STRUCTURES, SO IN A JUST WORLD, THEY SOMEHOW FIT INTO THE UNIV. ALG. PICTURE.

HOWEVER: $X^n \xrightarrow{*} \wp(X)$ IS NOT AN n-ARY OPERATION ON X!

(WRONG TYPE FOR A MOR. IN $\mathcal{T}$.)

THE PROBLEM

FACT. FOR SETS X, Y,

SupLat MOR. IN EACH VARIABLE

$$\mathrm{Hom}_{\mathrm{Set}}(Y^n, \mathcal{P}(X)) \cong \mathrm{Mult}_{\mathrm{SupLat}}(\mathcal{P}(Y)^n, \mathcal{P}(X))$$
$$= \mathrm{Mult}_{\mathrm{CABA}}(\mathcal{P}(Y)^n, \mathcal{P}(X)).$$

i.e., THERE IS AN n-ARY HYPEROPERATION ON $X \in \mathrm{Set}$ $\iff$ THERE IS AN n-ARY OPERATION ON $\mathcal{P}(X) \in \mathrm{CABA}$.

COMPLETE ATOMISTIC BOOLEAN ALGEBRAS

THE PROBLEM

THE GOAL BECOMES: SHOW THAT HMag, HSemi, HMon, Msc ARE ENRICHED-ALGEBRAIC OVER CABA.

THIS GIVES AN ALTERNATIVE PROOF OF THE PROPERTIES WE KNOW, PLUS REALIZATIONS OF THESE AS FINITARY ENRICHED MONADS, etc.

(ALGEBRAIC CAT'S ARE VERY NICE!)

THE SOLUTION

PROGRESS SO FAR:

1. Suplat IS KNOWN TO BE SYMMETRIC MONOIDAL CLOSED, BUT CABA $\subseteq$ Suplat IS NOT.

THEOREM. (R.) THE CLOSED SYMMETRIC MONOIDAL STRUCTURE ON Suplat RESTRICTS TO CABA.

"CAN ENRICH OVER CABA."

THE SOLUTION

2 WRITE DOWN τ_{HMag}, τ_{HSemi}, τ_{HMon}, τ_{Msc}.

HMag, HSemi, HMon ARE STRAIGHT FORWARD:

THE SOLUTION

2 <u>THEOREM</u>. (R.) THE CATEGORIES HMag, HSemi, HMon ARE ALGEBRAIC OVER CABA.

PROOF. IN EACH CASE, TAKE $\mathcal{T}$ TO BE THE FREE CABA-CATEGORY ON THE UNENRICHED THEORY.

THE SOLUTION

(2) MSC : USUAL DEFINITION CONTAINS THE FOLLOWING AXIOM —

(REVERSIBILITY) $\forall x, y, z \in (X, \star)$,

$$x \in y \star z \iff y \in x \star z^{-1}$$
$$\iff z \in y^{-1} \star x.$$

IT'S NOT OBVIOUS THAT THIS IS EQUATIONAL. NOTE IT'S EQUIVALENT TO

$$\chi_{y \star z}(x) = \chi_{x \star z^{-1}}(y) = \chi_{y^{-1} \star x}(z), \text{ SO...}$$

$(\forall x, y, z)$

THE SOLUTION

② **PROP.** (R.) A UNITAL HYPERMAGMA WITH INVERSES IS REVERSIBLE IFF

$$\begin{tikzcd}
X^3 \arrow[rr, "\blacktriangle"] \arrow[dr, "\alpha"] \arrow[ddr, "\beta"'] & & \{0,1\} \\
& X^3 \arrow[ur, "\blacktriangle"] & \\
& X^3 \arrow[uur, "\blacktriangle"'] &
\end{tikzcd}$$

COMMUTES.

NO TIME FOR DEF'S, BUT IT EXPRESSES EQUALITY OF INDICATOR FNS. I/T/O MONOIDAL STRUCTURE ON **Set**.

NOTE: $\{0,1\} \cong \mathcal{P}(\mathbb{1})$, SO THIS GIVES A DIAGRAM IN CABA.

THE SOLUTION

3 (TO-DO) VERIFY THAT DAY CONVOLUTION OF ENRICHED FUNCTORS COINCIDES WITH THE SYMMETRIC MONOIDAL CLOSED STRUCTURE ON HMag, HMon, Msc GIVEN IN

ELSEVIER
Journal of Algebra
Volume 676, 15 August 2025, Pages 408-474

Research Paper

Categories of hypermagmas, hypergroups, and related hyperstructures ☆

So Nakamura, Manuel L. Reyes

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FUTURE WORK

REAL QUICK: A MATROID IS A SET WITH A NOTION OF WHICH SUBSETS ARE LINEARLY INDEPENDENT.

- THERE IS A FUNCTOR

$$\text{Matroids} \longrightarrow \text{Msc.}$$

  WHAT SORT OF MOSAIC ARISES FROM A MATROID? (CHARACTERIZE THE ESSENTIAL IMAGE.)

FUTURE WORK

- CAN THESE RESULTS BE FORMULATED IN TERMS OF MOSAICS?

Matroids and quantum-secret-sharing schemes

Pradeep Sarvepalli* and Robert Raussendorf

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Phys. Rev. A **81**, 052333 – **Published 24 May, 2010**

ANY "IDENTICALLY SELF-DUAL" MATROID GIVES RISE TO A PURE-STATE QUANTUM S.S.S.

Local equivalence, surface-code states, and matroids

Pradeep Sarvepalli* and Robert Raussendorf

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Phys. Rev. A **82**, 022304 – **Published 5 August, 2010**

CSS CODE STATES ARE "MINOR CLOSED BINARY" MATROIDS

FUTURE WORK

- SYMPLECTIC MATROIDS GENERALIZE ORDINARY ONES. DO THEY ALSO EMBED INTO SOME CAT. OF HYPERS.?

Quantum codes and symplectic matroids

Publisher: IEEE

Cite This PDF

Pradeep Sarvepalli All Authors

CSS CODES "ARE" HOMOGENEOUS SYMPLECTIC MATROIDS.

FUTURE WORK

Appl Categor Struct (2018) 26:205–237
DOI 10.1007/s10485-017-9490-2

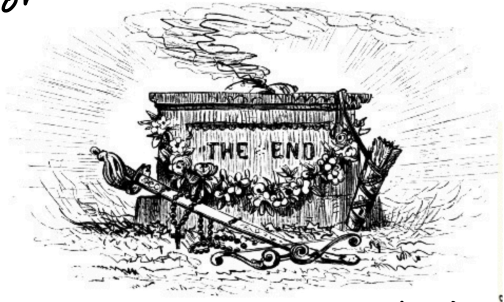
The Category of Matroids

Chris Heunen[1] · Vaia Patta[2]

- WE KNOW STUFF ABOUT Matroids, BUT NOTHING ABOUT THE CAT. OF SYMPLECTIC MATROIDS.

WHAT'S GOIN' ON IN THERE?

thanks for listening!



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